AP CALCULUS AB
TOPIC I: SETS AND FUNCTIONS

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1. Sets and Elements

A *set* is a collection of *elements*. The elements of a set are sometimes called *members* or *points*. We assume that we can distinguish between different elements, and that we can determine whether or not a given element is in a given set.

The relationship of two elements a and b being the same is *equality* and is denoted $a = b$. The negation of this relation is denoted $a \neq b$, that is, $a \neq b$ means that it is not the case that $a = b$.

The relationship of an element a being a member of a set A is *containment* and is denoted $a \in A$. The negation of this relation is denoted $b \notin A$, that is, $b \notin A$ means that it is not the case that $b \in A$.

A set is determined by the elements it contains. That is, two sets are considered equal if and only if they contain the same elements. We use the symbols "$\Rightarrow$" to mean "implies", and "$\Leftrightarrow$" to mean "if and only if". Then

$$A = B \quad \Leftrightarrow \quad (a \in A \Leftrightarrow a \in B);$$

in English, "A equals B if and only if (a is in A if and only if b is in B)".

We may describe a set by listing its members; such lists are surrounded by braces. For example the set of the first five prime integers is $\{2, 3, 5, 7, 11\}$. If a pattern is clear, we may use dots to indicate an infinite set; for example, to label the set of all prime numbers as P, we may write $P = \{2, 3, 5, 7, 11, 13, \dots\}$. The order of elements in a list is irrelevant in determining a set, for example, $\{5, 3, 7, 11, 2\} = \{2, 3, 5, 7, 11\}$. Also, there is no such thing as the "multiplicity" of an element in a set, for example $\{1, 3, 2, 2, 1\} = \{1, 2, 3\}$.

2. Subsets

If A and B are sets and all of the elements in A are also contained in B, we say that A is a *subset* of B or that A is *contained* in B and write $A \subset B$:

$$A \subset B \quad \Leftrightarrow \quad (a \in A \Rightarrow a \in B);$$

in English, "A is contained in B if and only (a is in A implies a is in B)". Every set is a subset of itself. We say that A is a *proper subset* of B is $A \subset B$ but $A \neq B$.

It follows immediately from the definition of subset that

$$A = B \quad \Leftrightarrow \quad (A \subset B \text{ and } B \subset A);$$

in English, "A equals B if and only if (A is a subset of B and B is a subset of A)."

A set containing no elements is called the *empty set* and is denoted $\varnothing$. Since a set is determined by its elements, there is only one empty set. Note that the empty set is a subset of any set.

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3. Set Operations

We may construct new sets as subsets of existing sets by specifying properties. Specifically, we may have a proposition $p(x)$ which is true for some elements x in a set X and not true for others. Then we may construct the set

$$\{x \in X \mid p(x) \text{ is true}\};$$

this is read "the set of x in X such that $p(x)$". The construction of this set is called *specification*. For example, if we let $\mathbb{Z}$ be the set of integers, the set P of all prime numbers could be specified as $P = \{n \in \mathbb{Z} \mid n \text{ is prime}\}$.

Let A and B be subsets of some "universal set" U and define the following set operations:

$$\begin{aligned} \text{Union:} \quad & A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\} \\ \text{Intersection:} \quad & A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\} \\ \text{Complement:} \quad & A \smallsetminus B = \{x \in U \mid x \in A \text{ and } x \notin B\} \end{aligned}$$

The pictures which correspond to these operations are called *Venn diagrams.*

Example 1. Let $A = \{1,3,5,7,9\}$, $B = \{1,2,3,4,5\}$. Then $A \cap B = \{1,3,5\}$, $A \cup B = \{1,2,3,4,5,7,9\}$, $A \smallsetminus B = \{7,9\}$, and $B \smallsetminus A = \{2,4\}$. $\square$

Example 2. Let A and B be two distinct nonparallel lines in a plane. We may consider A and B as sets of points. Their intersection is a set containing a single point, their union is a set consisting of all points on crossing lines, and the complement of A with respect to B is A minus the point of intersection. $\square$

If $A \cap B = \varnothing$, we say that A and B are *disjoint.*

The following properties are sometimes useful.

- $A = A \cup A = A \cap A$
- $\varnothing \cap A = \varnothing$ and $\varnothing \cup A = A$
- $A \subset B \Leftrightarrow A \cap B = A$
- $A \subset B \Leftrightarrow A \cup B = B$

The following properties state that union and intersection are commutative and associative operations, and that they distribute over each other. These properties are intuitively clear via Venn diagrams.

- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Since $(A \cap B) \cap C = A \cap (B \cap C)$, parentheses are useless and we write $A \cap B \cap C$. This extends to four sets, five sets, and so on. Similar remarks apply to unions.

The following properties of complement are known as *DeMorgan's Laws.* You should draw Venn diagrams of these situations to convince yourself that these properties are true.

- $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$
- $A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$

4. Cartesian Product

Let a and b be elements. The *ordered pair* with first coordinate a and second coordinate b consists of these two elements in the specified order. We denote this ordered pair by (a, b) and declare that it has the following "defining property":

$$(a, b) = (c, d) \quad \Leftrightarrow \quad (a = c \text{ and } b = d).$$

The ordered pair (a, a) is allowed, and $(a, b) = (b, a) \Leftrightarrow a = b$.

The *cartesian product* of the sets A and B is denoted $A \times B$ and is defined to be the set of all ordered pairs whose first coordinate is in A and whose second coordinate is in B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Example 3. Let $A = \{1, 3, 5\}$ and let $B = \{1, 4\}$. Then

$$A \times B = \{(1, 1), (1, 4), (3, 1), (3, 4), (5, 1), (5, 4)\}.$$

In particular, this set contains 6 elements. □

In general, if A contains m elements and B contains n elements, where m and n are natural numbers, then $A \times B$ contains mn elements. Consider the case where $A = B$; then $A \times A$ contains m^2 elements. We sometimes write A^2 to mean $A \times A$.

We have the following properties of cartesian products:

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

5. Numbers

The following familiar sets of numbers have standard names:

$$\text{Natural Numbers:} \qquad \mathbb{N} = \Big\{0, 1, 2, 3, \dots\Big\}$$

$$\text{Integers:} \qquad \mathbb{Z} = \Big\{\dots, -2, -1, 0, 1, 2, \dots\Big\}$$

$$\text{Rational Numbers:} \qquad \mathbb{Q} = \Big\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\Big\}$$

$$\text{Real Numbers:} \qquad \mathbb{R} = \Big\{\text{ numbers given by decimal expansions }\Big\}$$

$$\text{Complex Numbers:} \qquad \mathbb{C} = \Big\{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\Big\}$$

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Let $I \subset \mathbb{R}$. We say that I is *connected* if for every $a_1, a_2 \in A$ with $a_1 < a_2$, and every $x \in \mathbb{R}$, we have

$$a_1 < x < a_2 \quad \Rightarrow \quad x \in A.$$

In words, A is connected if for any two points in A, every point between these two points is also in A.

An *interval* is a connected set of real numbers which contains more than one element. There are nine types:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ (closed)
- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ (open)
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$ (closed)
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$ (open)
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$ (closed)
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$ (open)
- $(-\infty, \infty) = \mathbb{R}$

6. Functions

Let A and B be sets. A *function* from a set A to a set B is an assignment of every element in A to a unique element in B. Alternatively, a function is a method of sending each element of A to an element of B.

Let f be a function from A to B. If $a \in A$, the element of B to which a is assigned by f is denoted $f(a)$; in other words, the place in B to which a is sent by f is denoted $f(a)$. We declare that a function must satisfy the following "defining property":

$$\text{for every } a \in A \text{ there exists a unique } b \in B \text{ such that } f(a) = b.$$

If f is a function from A to B, this fact is denoted

$$f : A \to B.$$

We say that f *maps* A *into* B, and that f is a function *on* A. For this reason, functions are sometimes called *maps* or *mappings*. If $f(a) = b$, we say that a is *mapped to* b by f. We may indicate this by writing $a \mapsto b$.

Two functions $f : A \to B$ and $g : A \to B$ are considered *equal* if they act the same way on every element of A:

$$f = g \quad \Leftrightarrow \quad (a \in A \Rightarrow f(a) = g(a)).$$

Thus to show that two functions f and g are equal, select an arbitrary element $a \in A$ and show that $f(a) = g(a)$.

If A is sufficiently small, we may explicitly describe the function by listing the elements of A and where they go; for example, if $A = \{1, 2, 3\}$ and $B = \mathbb{R}$, a perfectly good function is described by $\{1 \mapsto 23.432, 2 \mapsto \pi, 3 \mapsto \sqrt{593}\}$.

However, if A is large, the functions which are easiest to understand are those which are specified by some *rule* or *algorithm*. The common functions of single variable calculus are of this nature.

Example 4. The following can be functions from $\mathbb{R}$ into $\mathbb{R}$:

- $f(x) = 0$;
- $f(x) = x$;
- $f(x) = x^3 + 3x + 17$.

The following can be functions from the set of positive real numbers into $\mathbb{R}$:

- $f(x) = \frac{1}{x}$;
- $f(x) = \sqrt{x}$.

Note that $\frac{1}{x}$ is not a function from $\mathbb{R}$ into $\mathbb{R}$, because it is not defined at $x = 0$. □

Some functions are constructed from existing functions by specifying cases.

Example 5. Let $\mathbb{R}$ be the set of real numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0; \\ x^3 - 1 & \text{if } x \geq 0. \end{cases}$$

Then, for example, $f(-2) = (-2)^2 + 2 = 6$ and $f(2) = 2^3 - 1 = 7$. □

Example 6. Let X be a set and let $A \subset X$. The *characteristic function* of A in X is a function $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 0 \text{ if } x \notin A; \\ 1 \text{ if } x \in A. \end{cases}$$

7. Images and Preimages

If $f : A \to B$, the set A is called the *domain* of the function and the set B is called the *codomain*. We often think of a function as taking the domain A and placing it in the codomain B. However, when it does so, we must realize that more than one element of A can be sent to a given element in B, and that there may be some elements in B to which no elements of A are sent.

If $a \in A$, the *image* of a under f is $f(a)$.

If $b \in B$, the *preimage* of b is a subset of A given by

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}.$$

If $C \subset A$, we define the *image* of C under f to be the set

$$f(C) = \{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$

The image of the domain is called the *range* of the function.

If $D \subset B$, we define the *preimage* of D under f to be the set

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}.$$

Notice that $f^{-1}(b)$ is not necessarily a singleton subset of A. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, then the preimage of the point 4 is

$$f^{-1}(4) = \{2, -2\}.$$

A function $f : A \to B$ is called *surjective* (or *onto*) if

$$\text{for every } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b.$$

Equivalently, f is surjective if $f(A) = B$. This says that every element in B is "hit" by some element from A.

A function $f : A \to B$ is called *injective* (or *one-to-one*) if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

Equivalently, f is injective if for all $b \in B$, $f^{-1}(b)$ contains at most one element in A.

A function $f : A \to B$ is called *bijective* if it is both injective and surjective. Such a function sets up a *correspondence* between the elements of A and the elements of B.

Example 7. First we consider "real-valued functions of a real variable". This simply means that the domain and the codomain of the function are subsets of $\mathbb{R}$.

- $f(x) = x^3$ is bijective;
- $g(x) = x^2$ is neither injective nor surjective;
- $h(x) = x^3 - 2x^2 - x + 2$ is surjective but not injective;
- $e(x) = 2^x$ is injective but not surjective.

Let $A = \{-1, 1, 2\}$. Some of the images and preimages of A are:

- $f(A) = \{-1, 1, 8\}$;
- $g(A) = \{1, 4\}$;
- $h(A) = \{0\}$;
- $f^{-1}(A) = \{-1, 0, \sqrt[3]{2}\}$;
- $g^{-1}(A) = \{-\sqrt[3]{2}, -1, 1, \sqrt[3]{2}\}$;
- $a^{-1}(A) = \varnothing$.

8. Composition of Functions

Let A, B, and C be sets and let $f : A \to B$ and $g : B \to C$. The *composition* of f and g is the function

$$g \circ f : A \to C$$

given by

$$g \circ f(a) = g(f(a)).$$

The domain of $g \circ f$ is A and the codomain is C. The range of $g \circ f$ is the image under g of the image under f of the domain of f.

Proposition 1. *Let $f : A \to B$ and $g : B \to C$ be surjective functions. Then $g \circ f : A \to C$ is an surjective function.*

Proposition 2. *Let $f : A \to B$ and $g : B \to C$ be injective functions. Then $g \circ f : A \to C$ is an injective function.*

Example 8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x - 9$. Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by $g \circ f(x) = x^2 - 9$ and $f \circ g : \mathbb{R} \to \mathbb{R}$ is given by $f \circ g(x) = x^2 - 6x + 9$. $\square$

This example demonstrates that composition of functions is not a commutative operation. However, the next proposition tells us that composition of functions is associative.

Proposition 3. *Let A, B, C, and D be sets and let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.*

9. Restrictions, Identities, and Inverses

Let $f : X \to Y$ be a function and let $Z = f(X)$ be the range of f. The same function f can be viewed as a function $f : X \to Z$. It is standard in this case to call the function, viewed in this way, by the same name. Note that the function $f : X \to Z$ is surjective. Thus any function is a surjective function onto its range.

Let $f : X \to Y$ be a function and let $A \subset X$ be a subset of the domain of f. The *restriction* of f to A is a function

$$f \restriction_A : A \to Y \text{ given by } f \restriction_A (a) = f(a).$$

Thus given any function and any subset of the domain, there is a function which coincides with the original one, but whose domain is the subset. For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ can certainly be viewed as a function on the integers, sending each integer to its square.

Let A be any set. The *identity function* on A if the function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$ for every $a \in A$. Thus the identity function on A is that function which does nothing to A. The identity function has the property that if $g : A \to C$, then $g \circ \mathrm{id}_A = g$, and if $h : D \to A$, then $\mathrm{id}_A \circ h = h$.

Let $f : A \to B$ be a function. We say that f is *invertible* if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In this case we call g the *inverse* of f. The inverse of a function f is often denoted f^{-1}.

If f is not injective, then f cannot be invertible. Sometimes we restrict the domain of f to a subset on which f is injective to invent a partial inverse.

10. Silly Examples

Example 9. Let C be the set of all the chickens in the world, and let H denote the set of all henhouses in the world. We wish to form a function

$$f : C \to H \text{ given by } f(\text{chicken}) = \text{ the henhouse that chicken lives in } .$$

- The domain of f is C, the set of all chickens. The codomain of f is H, the set of all henhouses. The range of f is the set of occupied henhouses.
- What can we say if some chickens are homeless? This would mean that f is not a function, since each member of the domain must be assigned to at least one element of the codomain.
- What can we say if some chickens live in more than one henhouse? Then again, this is not a function, since each member of the domain must be assigned to at most one element of the codomain.

Assume that each chicken lives in exactly one henhouse, so that f is a function.

- The image of the set of my chickens is the set of henhouses in which my chickens live.
- An out of control bioweapons producer in Southern Arkansas released a genetically engineered bacteria that infects all the henhouses in Nevada County (pronounced Nev Ay Da). We need to recall all of the chickens which live in those henhouses. The set of recalled chickens is the preimage of the set of infected henhouses.

Example 10. Let P equal the set of good college seniors, where "good" is defined as being good enough to be drafted by an NFL team. Let T be the set of NFL teams. Define a function

$$f : P \to T \quad \text{ given by } \quad f(\text{player}) = \text{ the team which drafted the player } .$$

The domain of f is the set of good college seniors, and the codomain of f is the set of all NFL teams. The range of f is the set of NFL teams which drafted at least one player.

- Let A denote the set of good seniors who attended Alabama. Then the image of A, $f(A)$, is the set of NFL teams who drafted a player from Alabama.
- Let $B = \{\texttt{Dolphins}, \texttt{Jaguars}, \texttt{Buccaneers}\}$. Then the preimage of B, $f^{-1}(B)$, is the set of good seniors who were drafted by a team from Florida.
- What would it mean if f were injective? It would mean that each team drafted at most one player.
- What would it mean if f were surjective?

Let C denote the set of cities in the United States, and define a function

$$g : T \to C \quad \text{ given by } \quad g(\text{team}) = \text{ the home city of the team } .$$

Let $h = f \circ g$.

- Let $p \in P$. Then $h(p)$ is the city to which player p is expected to move, after the draft.
- Let F denote the set of cities in Florida.
  Then $g^{-1}(F) = \{\texttt{Dolphins}, \texttt{Jaguars}, \texttt{Buccaneers}\}$, and $h^{-1}(F)$ is the set of good seniors who were drafted by a team from Florida.

11. Exercises

Exercise 1. Let $A = \{4, 5, 6, 7, 8, 9, 10, 11\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$, and $C = \{3, 6, 9, 12, 15, 18, 21\}$. Find the indicated set.

(a) $(A \cap B) \setminus C$
(b) $A \setminus (B \cup C)$
(c) $(A \setminus B) \cup C$

Exercise 2. Let A, B, and C be the following subsets of $\mathbb{N}$:

- $A = \{n \in \mathbb{N} \mid n \leq 25\}$;
- $E = \{n \in A \mid n \text{ is even}\}$;
- $O = \{n \in A \mid n \text{ is odd}\}$;
- $P = \{n \in A \mid n \text{ is prime}\}$;
- $S = \{n \in A \mid n \text{ is a square}\}$;

Compute the following sets.

(a) $(P \cup S) \cap O$
(b) $(E \setminus S) \cup P$
(c) $(O \cap S) \times (E \cap S)$

Exercise 3. Let $A = [0, 5]$, $B = (2, 7)$, $C = (6, 9)$, and $D = \{1, 3, 4, 7\}$. Find each of the following sets.

(a) $(A \cup B) \setminus D$
(b) $B \cup (C \cap D)$
(c) $A \setminus D$
(d) $(A \cup C) \setminus D$

Exercise 4. Let $A = \{x \in \mathbb{R} \mid -3 \leq x < 7\}$ and $B = \{x \in \mathbb{R} \mid 1 < x \leq 5\}$.
Find the indicated set.

(a) A
(b) B
(c) $A \cup B$
(d) $A \cap B$
(e) $A \setminus B$

Exercise 5. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9, 11\}$.
Find $C = (A \cup B) \setminus (A \cap B)$.

Exercise 6. Let $D = [2, 10]$ and $E = (\pi, 8]$. Find $F = (D \setminus E) \setminus \mathbb{Z}$.

Exercise 7. Sketch the graph of the set $[1, 3] \times ([1, 4] \setminus [2, 3])$ as a subset of $\mathbb{R}^2$.

Exercise 8. Sketch the graph of the set $([1, 5] \setminus (2, 4)) \times (\{1, 3\} \cup [4, 5])$.

Exercise 9. Let $A = [2, 3) \cup \{4\} \cup (5, 6]$. Sketch the graph of the set $A \times A$.

Exercise 10. Sketch the graph of the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 - 6x + y^2 - 4y \leq 0\}$.

Exercise 11. Draw Venn diagrams which demonstrate the following equations.

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(c) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Exercise 12. Let A and B be subsets of a set U. The *symmetric difference* of A and B, denoted $A \triangle B$, is the set of points in U which are in either A or B but not in both.

- **(a)** Draw a Venn diagram describing $A \triangle B$.
- **(b)** Find two set expressions which could be used to define $A \triangle B$. These expressions may use A, B, union, intersection, complement, and parentheses,

Exercise 13. Find the domain of the function $f(x) = \frac{\sqrt{x^2-3x-70}}{x^2-64}$. Express your answer in interval notation.

Exercise 14. Find the range of the function $g(x) = x^2 - 4x + 17$. Express your answer in interval notation.

Exercise 15. Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{Z}$ be the integers. Find examples of functions $f : \mathbb{Z} \to \mathbb{N}$ such that:
(a) f is bijective;
(b) f is injective but not surjective;
(c) f is surjective but not injective;
(d) f is neither injective nor surjective.

Exercise 16. Let $\mathbb{N}$ be the set of natural numbers. Let $A = [50, 70] \cap \mathbb{N}$. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = 3n$. Note that A is in both the domain and the codomain of f.
(a) Find the image $f(A)$.
(b) Find the preimage $f^{-1}(A)$.
(c) Is f injective? Is f surjective?

Exercise 17. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 6x^2 + 11x - 3$. Find $f^{-1}(3)$.

Exercise 18. We would like to define a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by $(p, q) \mapsto \frac{p}{q}$. Unfortunately, this does not make sense. Fix the problem, so that the resulting function is surjective but not injective.

Exercise 19. We would like to define a function $f : \mathbb{Q} \to \mathbb{Z}$ by $\frac{p}{q} \mapsto pq$. Unfortunately, this is not "well-defined". Figure out what this means and fix the problem. Is the resulting function injective?

Exercise 20. Let $f : X \to Y$ be a function and let $A, B \subset X$ and $C, D \subset Y$. Which of the following statements are true? If the statement is false, attempt to construct a counterexample.

- **(a)** $f(A \cup B) \subset f(A) \cup f(B)$
- **(b)** $f(A \cup B) = f(A) \cup f(B)$
- **(c)** $f(A \cap B) \subset f(A) \cap f(B)$
- **(d)** $f(A \cap B) = f(A) \cap f(B)$
- **(e)** $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- **(f)** $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Exercise 21. Let $f : X \to Y$ be a function. Which of the following statements are true?

- **(a)** f is surjective if and only if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.
- **(b)** f is injective if and only if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

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